Exam Advanced Mechanics, Wednesday, January 24, 2018

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5 problems (total of 50 points).
The solution of every problem on a separate piece of paper with name and student number.
Some useful formulas are listed at the end.

Problem 1 (10 pnts in total)

Find the shortest path on a surface given by $z = x^{3/2}$ between the two points $P_1 = (0, 0, 0)$ and $P_2 = (1, 1, 1)$.

3 pnts a. Express the path length as an integral along the curve.

2 pnts b. Give the resulting Euler equation(s) for the path.

3 pnts c. Show that the solution of the Euler equation can be written as $y(x) = A(1 + \frac{9x}{4})^{\frac{3}{2}} + B$.

2 pnts d. Determine A and B for the path between P_1 and P_2 (do not expect round numbers).

Problem 2 (9 pnts in total)

Given is a system of three masses, $m_1 = m$ at $\vec{r_1} = (2b, 0, 0)$, $m_2 = m$ at $\vec{r_2} = (0, 2b, 0)$, and $m_3 = 2m$ at $\vec{r_3} = (-b, -b, 0)$, which are connected by rigid, massless rods.

4 pnts a. Determine the inertial tensor, $\{I\}$, for this system for rotations around the origin.

2 pnts b. The system is rotating with $\vec{\omega} = \omega_0(1, 1, 0)$ determine $\vec{L}$ and $\vec{N} = d\vec{L}/dt$.

3 pnts c. The system is rotating with $\vec{\omega} = \omega_0(1, 0, 0)$ determine $\vec{L}$ and $\vec{N} = d\vec{L}/dt$.

Problem 3 (9 pnts in total)

A disk of mass M and radius R ($I = M_d R^2/2$) rolls without slipping down an inclined slope that makes an angle α with the horizontal. The slope itself can move horizontally without friction and has a mass m_s.

1 pnts a. Make a clear diagram in which you indicate the variables you will use in solving the problem.

4 pnts b. Construct the Lagrangian for this problem.

2 pnts c. Construct the equations of motion for this problem, DO NOT SOLVE!.

2 pnts d. Determine the constants of motion.

Problem 4 new (6 pnts in total)

Given is the Lorentz transformation

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

such that $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$, with $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$.

4 pnts a. Check by explicit construction that $(\Lambda)^{\alpha}{}_{\nu}(\Lambda)_{\alpha}{}^{\mu} = g^{\mu}_{\nu}$.

2 pnts b. Prove that $\Lambda^{\mu}{}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}}$.

Problem 5 new (16 pnts in total)

Given is the Lagrangian density for the electromagnetic field

$$\mathcal{L} = -mc\frac{\delta^3(\vec{x}-\vec{a})}{\gamma} - \frac{1}{c^2}A^{\mu}(x)j_{\mu}(x) - \frac{1}{16\pi c}F^{\mu\nu}F_{\mu\nu} \,.$$

The Euler-Lagrange equation for fields $\partial_{\mu}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}A_{\nu})} - \frac{\partial\mathcal{L}}{\partial A_{\nu}} = 0$ can be reduced to $\partial_{\mu}F^{\mu\nu} = \frac{4\pi}{c}j^{\nu}$ by introducing the electromagnetic field tensor $F^{\mu\nu} = \partial_{\mu}A^{\nu} - \partial_{\nu}A^{\mu}$.* We define the radiation field as $(A_r)^{\nu}(x) = S^{\nu}e^{i(x^{\mu}k_{\mu})}$ with S^{ν} a constant four vector, $k^{\mu} = [\omega/c, \vec{k}]$ and $x^{\mu} = [ct, \vec{x}]$.

2 pnts a. Show that $\tilde{F}^{\mu\nu} = F^{\mu\nu}$ for a gauge transformation $\tilde{A}^{\mu}(x) = A^{\mu}(x) + \partial^{\mu}G(x)$.

2 pnts b. Calculate $\frac{\partial\mathcal{L}}{\partial A_{\nu}}$.

2 pnts c. Calculate $\frac{\partial F_{\rho\sigma}}{\partial(\partial_{\mu}A_{\nu})}$

1 pnts d. Show that $\partial_{\mu}F^{\mu\nu} = \partial^2 A^{\nu}$ in the Lorentz gauge where $\partial_{\mu}A^{\mu} = 0$.

1 pnts e. Show that $x^{\mu}k_{\mu} = \omega t - \vec{k}\cdot\vec{x}$.

2 pnts f. Show that $(A_r)^{\nu}(x)$ is a solution of $\partial^2(A_r)^{\nu} = 0$ when $k^{\mu}k_{\mu} = 0$.

2 pnts g. Show that $(A_r)^{\nu}(x)$ obeys the Lorentz gauge when $k^{\mu}S_{\mu} = 0$.

2 pnts h. Take $k^{\mu} = [\omega/c, 0, 0, \omega/c]$ and $S^{\mu} = [0, 0, 1, 0]$ in the expression for $(A_r)^{\nu}(x)$ and calculate the electric field $\vec{E}$.

2 pnts i. Take $k^{\mu} = [\omega/c, 0, 0, \omega/c] = S^{\mu}$ in the expression for $A^{\nu}(x)$ and calculate the magnetic field $\vec{B}$. Does the answer surprise you?

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*$F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$

Possibly useful formulas:

The response of a damped oscillator $\ddot{x} + 2\beta\dot{x} + \omega_r^2 x = F(t)/m$ to a delta force at $t = 0$ is $\frac{1}{\omega_1 m} e^{-\beta t} \sin\omega_1 t$ for $t > 0$, where $\omega_1 = \sqrt{\omega_r^2 - \beta^2}$.

The Inertial tensor, $\{I\} = \sum_\alpha m_\alpha \left[\delta_{ij}\vec{r}_\alpha^{\,2}) - \vec{r}_{\alpha,i}\vec{r}_{\alpha,j}\right]$

$\vec{F}_B = \vec{F}_{\text{inert}} - 2m\vec{\omega} \times \vec{v}_B - m\dot{\vec{\omega}} \times \vec{r}_B - m\vec{\omega} \times (\vec{\omega} \times \vec{r}_B)$, and $\vec{v}_I = \vec{v}_B + \vec{\omega} \times \vec{r}_B$

$\vec{B} = \vec{\nabla} \times \vec{A}$; $\quad \vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial ct}$

$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$; $\quad \cos(\alpha - \beta) = \sin\alpha\sin\beta + \cos\alpha\cos\beta$

$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\,\vec{b} - (\vec{a} \cdot \vec{b})\,\vec{c}$

Integrals

For $c > 0$ we have:

$$\int e^{cx} dx = \frac{1}{c} e^{cx} \; ; \quad \int x\, e^{cx} dx = \frac{cx - 1}{c^2} e^{cx} \; ; \quad \int x^2 e^{cx} dx = \frac{c^2x^2 - 2cx + 2}{c^3} e^{cx}$$